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Self-oscillation in a detuned cavity

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A simple symmetry relates optical bistability in a ring cavity tuned near resonance and multimode instability and self-oscillation in a cavity excited midway between resonances.

It has been predicted that the continuous wave (c.w.) output of a bistable cavity may become unstable to multimode self-oscillation. Such multimode instabilities were first discussed by Bonifacio & Lugiato (1978) for absorptive bistability in a ring cavity. Their analysis was later extended by Lugiato (1980) to dispersive bistability. Recently, much attention has been given to this subject following Ikeda's identification of an instability leading to period-doubling and chaos in dispersive bistability (Ikeda 1979; Ikeda *et al.* 1980).

A detailed study of the work done by Ikeda led myself and co-workers to discover a second multimode instability for a saturable absorber in a ring cavity (Carmichael *et al.* 1982). A telling distinction exists between this and the instability studied by Bonifacio & Lugiato. They studied absorptive bistability in a resonant cavity. The instability that we have discovered occurs with the injected laser and resonant absorber tuned midway between cavity resonances. In a high-finesse cavity the Ikeda instability behaves similarly (Firth 1981; Carmichael *et al.* 1982; Bar-Joseph & Silberberg 1983). These observations provide the clue to the central result of this paper; the stability analysis for a nonlinear ring cavity exhibits a symmetry that establishes a one-to-one correspondence between optical bistability in a cavity tuned near resonance and multimode self-oscillation in a cavity tuned between resonances. The theory of absorptive and dispersive bistability can then be transferred as a whole to the description of corresponding multimode instabilities in a cavity tuned between resonances.

For simplicity I consider the plane-wave theory of absorptive bistability for a two-level homogeneously broadened medium in a ring cavity, and give detailed results only for the mean-field limit. My central conclusions are, however, quite general. They hold for dispersive bistability, for a gaussian-mode theory, and beyond the mean-field limit. On the other hand, they do not hold (at least not without qualification) in a standing-wave cavity, although it must be recognized that multimode instabilities have been predicted there also (Casagrande *et al.* 1980; Firth 1981).

The general stability analysis for a ring cavity containing a two-level homogeneously broadened absorber gives the following characteristic equation for eigenvalues λ governing the linearized dynamics (Carmichael 1983):

$$1+R^2\,\mathrm{e}^{-2\lambda\tau}\left[\frac{E(L)}{E(0)}\right]^{2/(1+\lambda T_2)}\left[\frac{(1+\lambda T_1)\,(1+\lambda T_2)+E(0)^2}{(1+\lambda T_1)\,(1+\lambda T_2)+E(L)^2}\right]^{\frac{1}{2}(2+\lambda T_2)/(1+\lambda T_2)}$$
$$-R\,\mathrm{e}^{-\lambda\tau}\left[\frac{E(L)}{E(0)}\right]^{1/(1+\lambda T_2)}\left\{1+\left[\frac{(1+\lambda T_1)\,(1+\lambda T_2)+E(0)^2}{(1+\lambda T_1)\,(1+\lambda T_2)+E(L)^2}\right]^{\frac{1}{2}(2+\lambda T_2)/(1+\lambda T_2)}\right\}\cos\theta=0. \quad (1)$$

Here R is the mirror reflection coefficient, T_1 and T_2 are atomic relaxation times, τ is the cavity round-trip time, θ is the cavity detuning $(-\pi \leqslant \theta < \pi)$ and $E(0)$ and $E(L)$ are dimensionless field amplitudes at either end of the medium. Solutions to (1) with $\text{Re}\,(\lambda) = 0$ define instability boundaries where a stable mode, $\text{Re}\,(\lambda) < 0$, becomes unstable, $\text{Re}\,(\lambda) > 0$, as system parameters are varied. Switching points in absorptive bistability $(\theta = 0)$ are defined by the requirement

$$1+R^2\frac{E(L)^2}{E(0)^2}\frac{1+E(0)^2}{1+E(L)^2}-R\frac{E(L)}{E(0)}\left[1+\frac{1+E(0)^2}{1+E(L)^2}\right]=0, \tag{2}$$

where, if (2) is satisfied, (1) has a solution $\lambda = 0$, indicating marginal stability for the resonant cavity mode. In the limit $\lambda T_1 \to 0$, $\lambda T_2 \to 0$, (1) is a function of $\exp(\lambda\tau)$ alone and all the cavity modes become unstable at the bistable switching points, i.e. when (2) is satisfied, (1) has solutions $\lambda_n = \mathrm{i}n\pi/\tau$, $n = 0, \pm 2, \pm 4, \ldots$, where $\text{Re}\,(\lambda_n) = 0$ for every n and $\text{Im}\,(\lambda_n)$ identifies the cavity mode frequencies measured with respect to the *resonant* laser frequency (see figure 1). Observe now, that, with $\lambda T_1 = \lambda T_2 = 0$, (1) is invariant under the transformation $\lambda \to \lambda + \mathrm{i}\pi/\tau$, $\theta \to \theta + \pi$. It follows that all cavity modes become unstable at the same instability boundaries (defined by (2)) in a cavity tuned midway between resonances $(\theta = -\pi)$. Now (1) has solutions $\lambda_n = \mathrm{i}n\pi$, $n = \pm 1, 3, \ldots$, where $\text{Im}\,(\lambda_n)$ identifies cavity mode frequencies measured with respect to the *detuned* laser frequency (see figure 1).

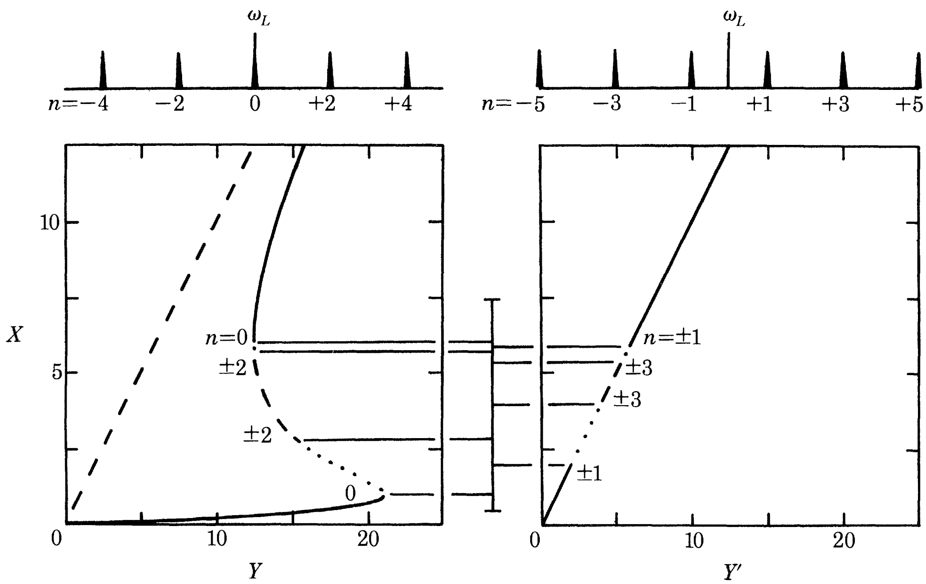

FIGURE 1. Multimode instabilities for $C = 20$, $T_1/\tau = 1.0$ and $T_1/T_2 \gg 1$. To the left is a plot of the state equation for absorptive bistability $(\theta = 0)$ and to the right the state equation for the corresponding detuned system $(\theta = -\pi)$. In dotted regions one mode is unstable and in regions of broken lines two modes are unstable.

An exact correspondence between the range of bistability $(\theta = 0)$ and the range of instability in a detuned cavity $(\theta = -\pi)$ exists only for $T_1/\tau \to 0$, $T_2/\tau \to 0$. I will illustrate the situation for finite T_1/τ and T_2/τ with explicit results for the mean-field limit $(1-R) \ll 1$, $\alpha L \ll 1$, with $C = \alpha L/4(1-R)$, where α is the resonant absorption coefficient. With $E(L) = X$ and $E(0) = X[1+(1-R)\,2C/(1+X^2)]$, perturbative solutions to (1) yield

$$\text{Re}\,(\lambda_n) = -\tau^{-1}(1-R)\,\text{Re}\left[1+\frac{2C}{1+X^2}\,\frac{1-X^2+\mathrm{i}n\pi T_1/\tau}{(1+\mathrm{i}n\pi T_1/\tau)\,(1+\mathrm{i}n\pi T_2/\tau)+X^2}\right], \tag{3}$$

with $n = 0, \pm 2, \pm 4, \ldots$ for $\theta = 0$, and $n = \pm 1, 3, \ldots$ for $\theta = -\pi$. In figure 2 the boundaries of instability $\mathrm{Re}\,(\lambda_n) = 0$ are plotted as a function of T_1/τ and X for $n = 0, \pm 2, \pm 4$ in absorptive bistability, and $n = \pm 1, \pm 3, \pm 5$ in the corresponding detuned cavity. The range of X between the vertical lines labelled $n = 0$ is the range of the negative slope branch in the bistable system. Here the resonant mode is unstable for all T_1/τ. The non-resonant modes are unstable whenever T_1/τ and X define a point lying under, or inside, the plotted curves. In figure 2(*a*) the horizontal bar follows successive changes of stability as a function of X in a system with $T_1/\tau = 1.0$, $T_1/T_2 \gg 1$. In figure 1 these changes of stability are displayed on the respective steady-state curves:

$$Y = X[1 + 2C/(1 + X^2)] \tag{4}$$

for $\theta = 0$, and

$$Y' = X \tag{5}$$

for $\theta = -\pi$. Here Y and Y' are dimensionless input field amplitudes, with $Y = (1-R)^{-1}\,(2\mu/\hbar)\,(T_1\,T_2)^{\frac{1}{2}}\,E_{\mathrm{i}}$ and $Y' = \frac{1}{2}(2\mu/\hbar)\,(T_1\,T_2)^{\frac{1}{2}}\,E_{\mathrm{i}}$, where μ is the atomic dipole moment.

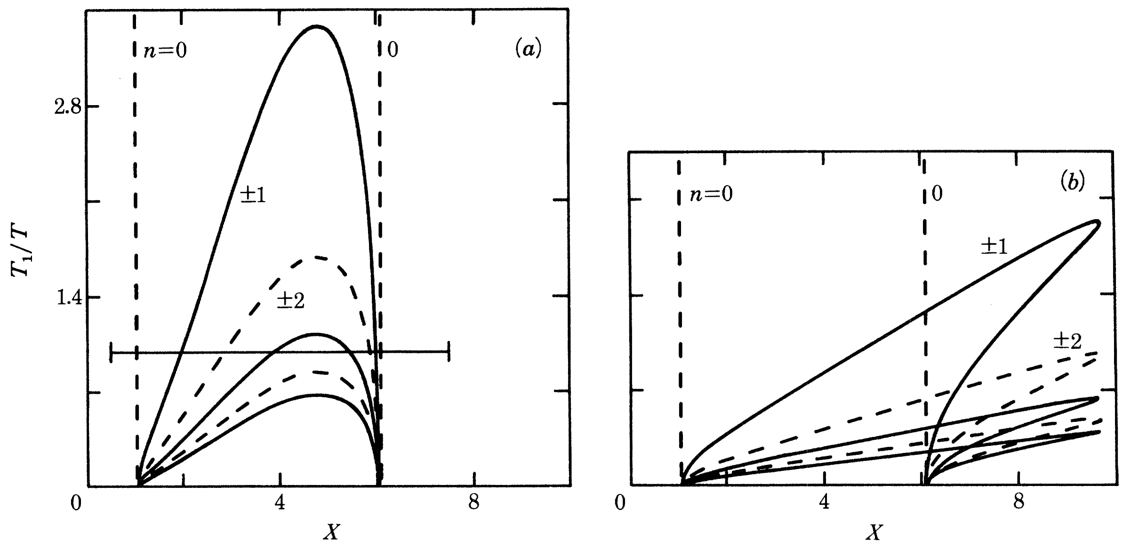

FIGURE 2. Instability boundaries for $C = 20$ and (*a*) $T_1/T_2 \gg 1$; (*b*) $T_1/T_2 = 0.5$. Broken curves are for $\theta = 0$ and solid curves are for $\theta = -\pi$.

In figure 2*b* the stability boundaries are distorted so that they extend outside the range of the resonant mode instability. This is the basis for self-pulsing instability along the upper branch in absorptive bistability, as studied by Bonifacio & Lugiato (1978). The only significance it holds in the corresponding detuned cavity is that the $n = \pm 3$ modes, for example, might be unstable, while the $n = \pm 1$ modes remain stable. This crossing of the instability boundaries does not occur if $T_1/T_2 \gtrsim C$ ($C \gg 1$), as in figure 2*a*. It is also eliminated in a gaussian-mode theory (Lugiato & Milani 1983).

If we return to the limit $T_1/\tau \rightarrow 0$, $T_2/\tau \rightarrow 0$, the relation between bistability and self-oscillation in a detuned cavity becomes even closer when we consider the form that this oscillation takes. In this limit cavity dynamics can be modelled by a nonlinear map (as in Ikeda (1979)). For the mean-field limit and $\theta = -\pi$.

$$X_{n+1} = 2Y' - X_n\{1 - (1-R)\,[1 + 2C/(1 + X_n^2)]\}. \tag{6}$$

The fixed point $X = Y'$ is unstable for $|dX_{n+1}/dX_n| > 1$. This is equivalent to the condition $dY/dX < 0$ obtained from (4), from which my central conclusion again follows. Now if we look for a two-cycle, an oscillation between X_1 and X_2, to replace the unstable fixed point, this requires

$$X_1 + X_2 = 2Y',$$
$$X_1\left(1+\frac{2C}{1+X_1^2}\right) = X_2\left(1+\frac{2C}{1+X_2^2}\right). \tag{7}$$

It follows that X_1 and X_2 are a pair of states satisfying the state equation (4) for some Y, a function of Y'. Figure 3 shows these oscillatory states in detail. For $X_a \leqslant Y' \leqslant Y'_b = \frac{1}{2}(X_b + \tilde{X}_b)$ there is a stable oscillation between the middle and lower branches of the corresponding bistability curve. For $Y'_b \leqslant Y' < Y'_a \simeq \frac{1}{2}(X_a + \tilde{X}_a)$ there is a stable oscillation between the upper and lower branches of the corresponding bistability curve. For $X_b < Y' < Y'_a$ there is an unstable oscillation between the upper and middle branches of the corresponding bistability curve.

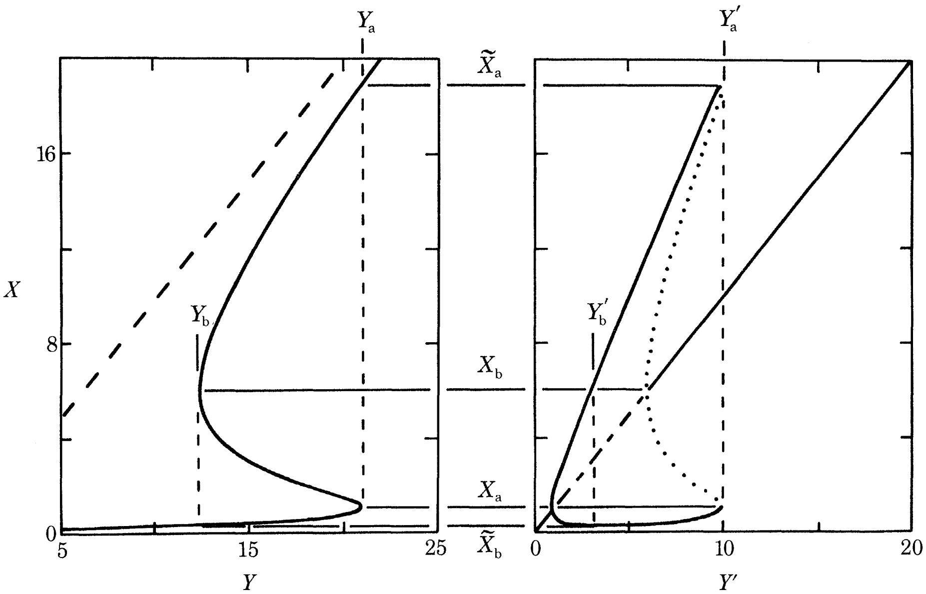

FIGURE 3. The relation between oscillation in a detuned cavity ($\theta = -\pi$), to the right, and the steady states of the corresponding bistable system ($\theta = 0$), to the left, for $C = 20$. Fixed points along the broken portion of $X = Y'$ are unstable. The solid branches bifurcating from $Y' = X_a$ are the states X_1 and X_2 of stable two-cycles. The dotted branches bifurcating from $Y' = X_b$ correspond to unstable two-cycles.

To summarize, for every example of bistability in a nonlinear ring cavity tuned near resonance, there exists a corresponding multimode instability leading to self-oscillation between states of the bistability curve, in a cavity tuned between resonances. This correspondence is one-to-one in the limit $T_1/\tau \rightarrow 0$, $T_2/\tau \rightarrow 0$, and identifies a range of multimode instability that extends to finite T_1/τ, T_2/τ.

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